MMP Learning Seminar

Week 95:

Content:

Termination of pseudo-effective 4-fold flips.

Minimal Model Program:

- Existence of flips. ✓
- Termination of flips.
- Abundance conjecture. (K_X nef $\Longrightarrow$ K_X semiample).

Existence:

Mori 88: Existence of flips for terminal 3-folds

Shok 92: " " " klt 3-folds

Shok 96: " " " lc 3-folds.

Shok 06: " " " of flips for some 4-folds

BCHM 06: Existence of klt flips.

HX, Birkar 2010: Existence of lc flips.

Termination:

Shok 92, 96: Termination of 3-fold lc flips.

Fuj 04: Termination of flips for term/canonical 4-folds

BCHM 06: Termination of flips with scaling for general type klt pairs

Abundance:

Kaw: In dimension 3, the terminal case

McKernan - Keel: In dimension 3, the lc case.

MMP (log canonical)

dim 3	dim 4	dim 5	...
fully settled.	Missing:	Missing:	
	• Termination of flips when X is uniruled	• Termination	...
	• Abundance.	• Abundance.	...

Theorem (M, 2018): X klt proj 4-fold with K_X pseff.
Then any MMP for X terminates.

Generalized pairs:

(X, B, M) $\quad$ X normal proj.

$B \geq 0$

called b-nef divisor. $\leftarrow$ M is push-forward of nef.

birationally.

$K_X + B + M$ is $\mathbb{R}$-Cartier

$$\begin{tikzcd} X' \arrow[d, "\text{proj bir}"'] & \supseteq & M' = \sum_i \lambda_i M_i' \arrow[d] \\ X & \supseteq & M. \end{tikzcd}$$

M_i': nef Cartier

log discrepancies: $\quad Y \xrightarrow[\varphi]{\text{proj biv}} X \qquad \varphi^*(K_X + B + M) = K_Y + B_Y + M_Y$

$$a_E(X, B, M) = 1 - \mathrm{coeff}_E(B_Y).$$

(X,B,M) is gklt if $a_E(X,B,M) > 0$.

(X,B,M) is glc if $a_E(X,B,M) \geq 0$.

(X,B,M) is gdlt if it is glc & the glce are strata of $\lfloor B \rfloor$.

Example:

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed] & & X_{min}. \end{tikzcd}$$

$K_X + \lambda M \equiv (1+\lambda) K_X$

Then K_X is push-forward of nef

$$M = K_X = p_* \underbrace{q^*(K_{X_{min}})}_{\text{nef}}$$

X is the min model $\iff$ $\mathrm{glct}(X, M) = \infty$.

what is the number that we can write here so this is glc.

$(X, \lambda M)$ $\qquad K_X + M \sim_{\mathbb{Q}} K_X + K_X = 2K_X.$

Definition: Let M be a b-nef divisor.

X be a klt variety.

$$\mathrm{glct}(X; M) = \max \{ t \mid (X, tM) \text{ is glc} \}.$$

Remark: The less positive M is on X, the smaller these numbers get.

M is nef on X (descends on X) then $\mathrm{glct}(X; M) = \infty$.

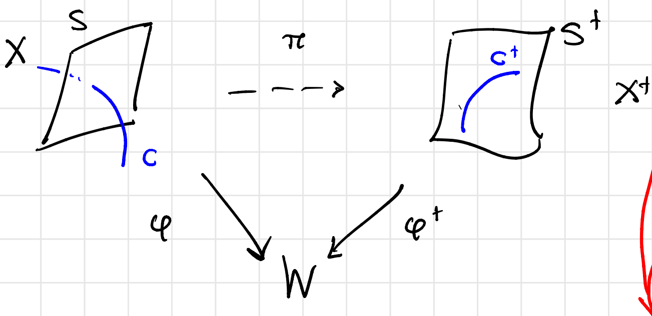
Theorem (Birkar-Zhang): M is b-nef divisor with Cartier index I. X klt projective n-dim variety.

Then $\mathrm{glct}(X, M)$ belongs an ACC set that only depends on n & I

Special Termination:

(X, B, M) gdlt pair $\qquad S = \lfloor B \rfloor = \text{glcc}(X, B, M)$.

Let $X \dashrightarrow X^+$ is an step of the $(K_X + B + M)$-MMP that intersects S.



Flip:

- π is (K_X+B+M)-neg.
- π is small.
- $\rho(X/W) = \rho(X^+/W) = 1$
- (K_X+B+M) is antiample over W & $(K_{X^+}+B^++M^+)$ ample over W.

These define a quasi-flip.

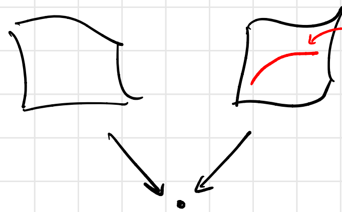
Lemma 1: The induced birational map.

$$(S, B_S, M_S) \dashrightarrow (S^+, B_{S^+}, M_{S^+}).$$

is a quasi-flip.

Proof: Follows from adjunction & negativity Lemma.

Lemma 2: Assume termination of flips in dim n, Then any sequence of quasi-flips in dim n under a DCC set terminates.



coefficient of the divisors that we extract belongs to a DCC set.

Proof: Using the condition on the DCC one can show that qflips terminate in codimension 1.

$$\begin{tikzcd}
 & S_1 \arrow[r, dashed] \arrow[d] \arrow[dr, "\text{small}"'] & S_2 \arrow[r, dashed] \arrow[d] & \cdots \arrow[dr, dashed] & \\
(S, B_S, M_S) \arrow[ur, dashed] \arrow[r] \arrow[drr] & W_1 & W_2 & & (S^+, B_{S^+}, M_{S^+}) \arrow[dll] \\
 & & W & &
\end{tikzcd}$$

$-(K_S + B_S + M_S)$ ample over W.

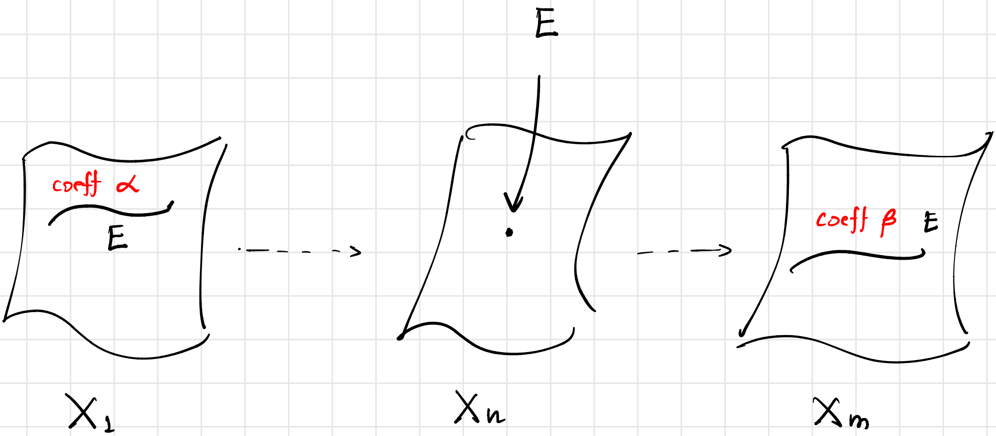
$K_{S^+} + B_{S^+} + M_{S^+}$ ample over W

Run MMP for $K_S + B_S + M_S$ over W.

Special Termination.

$\boxed{n=4}$

Proposition 1: Assume termination of flips in dimension $\boxed{3}$ $n-1$.
Let (X, B, M) be a gen dlt pair of dim n.
Then, any MMP for $(X, B+M)$ is eventually disjoint from $\lfloor B \rfloor$. $S =$



$m > n$

$0 < \beta < \alpha$

Remark: The # of log discrepancies in [0, 1] is finite for a gklt pair.

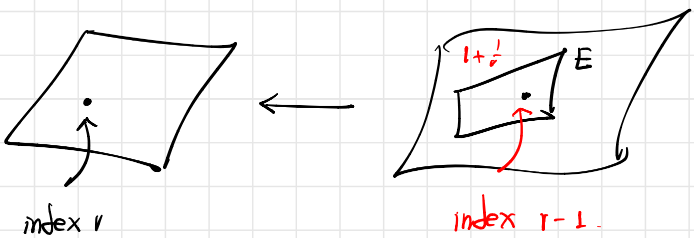
3-fold sing:

rK_X Cartier.

$(X; x)$ terminal 3-fold sing. of index r.

$\pi_1^{loc}(X; x) \simeq \mathbb{Z}/r\mathbb{Z}$.

Theorem (Chen, 90's): Let $(X; x)$ be a terminal 3-fold sing of index r. There exists weighted blow-up extracting a divisor with log discrepancy $1 + \frac{1}{r}$.



$$\begin{tikzcd} Y \arrow[d] & \text{term.} \\ X \end{tikzcd}$$

Corollary: $(X; x)$ terminal 3-fold sing

\# of divisors with log discrepancy in $(0, 1+\varepsilon)$ is $< N$.

$(1, 1+\varepsilon)$

Then $|\pi_1^{loc}(X; x)| < f(\varepsilon, N)$.

Theorem (Shokurov): $(X; x)$ klt 3-fold sing

\# of divisors with log discrepancy in $(\varepsilon, 1+\varepsilon)$ is $< N$.

Then $|\pi_1^{loc}(X; x)| < f(\varepsilon, N)$.

Remark: $(X; x)$ klt sing.

$$Y \xrightarrow{\varphi} X \quad \text{log resolution}$$

D $\mathbb{Q}$-Cartier through x.

Then, the index of D at x is controlled above by the largest denom of $\varphi^*(D)$.

(This is an application of Cone Theorem).

$m(\varphi^* D)$ Cartier $=$ Line bundle

$$m(\varphi^* D) \equiv_X 0$$

Proposition 2: (X, B, M) is 3-fold gen ε-klt (all log discrp $\geq \varepsilon$).

\# of log discrepancies in $(\varepsilon, \varepsilon+1) < N$.

Assume coeff B & coeff M $\in \mathcal{R}$ finite set.

Then, there exists a finite set $A(\mathcal{R}, \varepsilon, N)$

$\star$) $\quad a_E(X, B, M) \in [0,1] \Longrightarrow a_E(X, B, M) \in A$.

$\star\star$) $\quad a_E(X, B, M) \in [0,2] \Longrightarrow a_E(X, B, M) \in A$.

$\quad c_E(X)$ is a curve

Termination of flips for gklt 3-folds:

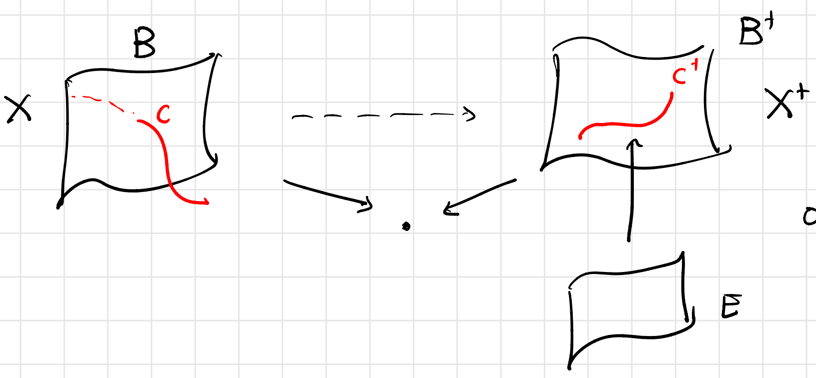
$(X, B, M) \dashrightarrow (X_1, B_1, M_1) \dashrightarrow \cdots$ sequence of flips

Step 1: Reduce to the $\mathbb{Q}$-factorial

Step 2: We control ε, N & R. $\Longrightarrow |\pi_1(X_i; x)| < f(\varepsilon, N)$ for any x & i.

Step 3: We reduce to the case in which (X, B, M) is terminal.

Step 4: Reduce to the case when $B = 0$.



X^+ is smooth along C^+

$a_E(X^+, B^+, M^+) \in [0, 2]$

$\Downarrow$

$a_E \in A$.

Step 5: Reduce to the case when $M = 0$.

We finally reduced to the terminal case with $B = M = 0$ $\square$

Theorem: Termination of pseff 4-fold flips.

Proof. We know that there is a minimal model X_{min}

$$\begin{array}{ccc}
\lambda_0 & X & \dashrightarrow \; X_{min} \\
\leq & \big\downarrow & \\
\lambda_1 & X_1 & \\
\leq & \big\downarrow & \\
\lambda_2 & X_2 & \\
\leq & \big\downarrow & \\
\lambda_3 & X_3 & \\
 & \big\downarrow & \\
 & \vdots &
\end{array}$$

$\lambda_i = \mathrm{glct}(X_i, M)$.

$M = p_* q^*(K_{X_{min}})$. index is fixed in the proof

Also a sequence of flips $(X_i, \lambda_i M)$.

$\lambda_i \leq \lambda_{i+1}$

It must stabilize with λ_∞.

4-dim.

$$(X, \lambda_\infty M) \dashrightarrow (X_1, \lambda_\infty M) \dashrightarrow \cdots \dashrightarrow (X_n, \lambda_\infty M) \dashrightarrow$$

glc not gklt.

gklt 3-dim term $\Longrightarrow$ special termination in dim 4.

this sequence is eventually disjoint from

$$\mathrm{glcc}(X_i, \lambda_\infty M) = W_i$$

$$\lambda_i' = \mathrm{glct}(X_i \setminus W_i, M) > \lambda_\infty.$$

$$\lambda_1' \leq \lambda_2' \leq \lambda_3' \leq \cdots \xrightarrow{\text{BZ16}} \lambda_\infty' > \lambda_\infty$$

$$\lambda_\infty < \lambda_\infty' < \lambda_\infty'' < \lambda_\infty''' < \cdots$$

will violate BZ16.

$\square$